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Mean relaxation time approximation for dynamical correlation functions in stochastic systems near instabilities

II. The single mode laser

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Correlation functions for the stochastic description of the single mode laser are investigated using the simple approximation method presented in part I [1]. The application of the mean relaxation time approximation to stochastic systems with state spaces of dimension $d > 1$ is thereby demonstrated. Our approach yields results which are indistinguishable from the actual lineshape. Hence, they constitute a slight improvement over the results of Risken and co-workers, where these results are based on an analysis of the lowest eigenvalue of the stochastic operator.

1. Introduction

In part I of the present series [1] we presented a simple approximation for dynamical correlation functions in stochastic systems, called the *mean relaxation time approximation* (MRTA). The MRTA is the lowest order approximation of a more general approach, the *generalized moment expansion method* [2] (GME), which is based on a simultaneous high- and low-frequency expansion of the exact correlation functions. The GME has been applied successfully to the description of observables for various stochastic transport processes, e.g., particle number correlation functions in reaction-diffusion systems [3, 4] and the Mößbauer absorption spectrum of Brownian particles modelling protein-internal motions [5–7]. Since the method takes into account also the low-frequency properties of the exact correlation functions, it is particularly suited for the correct description of the long-time behavior of these observables. The long-time behavior of correlation functions is of particular interest in systems near instabilities. At instabilities one expects slow relaxation processes (critical slowing down). Conventional methods based solely on high-frequency approximations, like the Mori-Zwanzig formalism [8, 9] (without additional refinements like the mode-coupling approach, see [10] and references cited therein), can give incorrect results in such a situation. In Part I we have demonstrated that already a lowest-order approximation, the MRTA, is able to describe correctly the critical slowing down of correlation functions in the Schlögl models, which describe autocatalytic chemical reaction systems exhibiting phase transitions.

Here we employ the MRTA to analyze correlation functions in the stochastic description of the single mode laser [10–14]. Due to the spontaneous emission processes in the laser medium, the electrical field amplitude of the laser radiation undergoes stochastic fluctuations [15]. These fluctuations, in particular the slow phase fluctuations, become relevant at and above the laser threshold, and are responsible for the actual spectral lineshape of the laser radiation.

The stochastic model for the single mode laser has been investigated extensively, see [10–14] and references cited therein. Our goal in this communication is

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(i) to demonstrate how the MRTA can be applied to stochastic systems with state spaces of dimension $d>1$ (the single mode laser is a generic example for such systems); and
(ii) to demonstrate that the known results for the single mode laser, which were previously determined mainly by eigenvalue analysis [11–14], can be determined alternatively by the MRTA in a coherent and (numerically) simpler way.

We will show that the MRTA obtains additional improvements over the known results for an effective single-Lorentzian description of the lineshape.

In the next section we shall treat the case of the single mode laser without detuning. Analytical and numerical results for the inverse effective linewidth, i.e. the mean relaxation time, of the intensity and field amplitude fluctuations are derived. In Sect. 3 a generalization of the model to the case of detuning of the atomic transition and resonance cavity frequencies is considered. To deal with this case a modification of the MRTA is presented, and effective linewidth and frequency shift are determined numerically. The results and possible extensions to other applications are discussed in Sect. 4. In an Appendix we show how the MRTA can be applied to one-dimensional system with multiplicative white noise.

2. Single mode laser

In the case of the single mode laser, stochastic fluctuations of the complex field amplitude E of the laser radiation are described by the Fokker-Planck equation [14]

$$\frac{\partial}{\partial t} p(E,t)=\mathbf{L}(E)\, p(E,t) \tag{2.1}$$

for the probability distribution $p(E,t)$, where $\mathbf{L}(E)$ denotes the Fokker-Planck operator

$$\mathbf{L}(E)=\frac{\partial}{\partial E^*}\left[\frac{\partial}{\partial E}-(a-|E|^2)\,E\right]. \tag{2.2}$$

The pump-parameter a is negative below and positive above the laser threshold. The Fokker-Planck operator (2.2) describes a stochastic system that obeys the principle of detailed balance [14], and the drift forces in (2.2) can be derived from a potential

$$U(E)=-\tfrac{1}{2}a|E|^2+\tfrac{1}{4}|E|^4. \tag{2.3}$$

Therefore, the stationary distribution of (2.1) is given by the Boltzmann distribution,

$$p_0(E)\propto\exp(-U(E)). \tag{2.4}$$

The spectral lineshape of the laser radiation is determined by the steady state autocorrelation functions of the *intensity fluctuations*,

$$C_I(t)=\langle \delta I(0)\, \delta I(t)\rangle, \tag{2.5a}$$

or the *field amplitude fluctuations*,

$$C_E(t)=\langle \delta E^*(0)\, \delta E(t)\rangle, \tag{2.5b}$$

depending on the experimental setup (see e.g. [16] and [17]. $\langle\ \rangle$ denotes average with respect to the stationary distribution $p_0(E)$ and has also the properties of an inner product on the space of functions, which will be used below. The intensity I is given by $I=|E|^2$, and intensity and field fluctuations are $\delta I=I-\langle I\rangle$ and $\delta E=E-\langle E\rangle$, respectively ($\langle E\rangle=0$ in our case).

In the MRTA the correlation functions (2.5) are approximated by a single exponential,

$$C_j(t)\approx C_j(0)\exp(-t/\tau_j), \quad j=I,E, \tag{2.6}$$

with $C_I(0)=\langle I^2\rangle-\langle I\rangle^2$, and $C_E(0)=\langle |E|^2\rangle=\langle I\rangle$. The mean relaxation times τ_I and τ_E in (2.6) are given by matrix elements of the *inverse* of the adjoint Fokker-Planck operator,

$$\mathbf{L}^+(E)=\left[\frac{\partial}{\partial E}+(a-|E|^2)\,E\right]\frac{\partial}{\partial E^*}, \tag{2.7}$$

through

$$\tau_I=-\langle \delta I[\mathbf{L}^+(E)]^{-1}\delta I\rangle/C_I(0), \tag{2.8a}$$

and

$$\tau_E=-\langle \delta E^*[\mathbf{L}^+(E)]^{-1}\delta E\rangle/C_E(0). \tag{2.8b}$$

This special form of approximation can be derived from a consideration of the Laplace-transported observable [1], e.g. in the case of $C_E(t)$,

$$\begin{aligned}\tilde{C}_E(\omega)&=\int_0^\infty e^{-\omega t}\, C_E(t)\\ &=\langle \delta E^*[\omega-\mathbf{L}^+(E)]^{-1}\delta E\rangle.\end{aligned} \tag{2.9}$$

A formal expansion of $\tilde{C}_E(\omega)$ for small frequencies gives

$$\tilde{C}_E(\omega)\sim\sum_{n=0}^{\infty}(-\omega)^n\mu_{-(n+1)}. \tag{2.10}$$

The expansion coefficients (*generalized moments*) μ_{-n} are given by matrix elements of powers of the inverse adjoint Fokker-Planck operator,

$$\mu_{-n}=(-1)^n\langle \delta E^*[\mathbf{L}^+(E)]^{-n}\delta E\rangle. \tag{2.11}$$

In order to correctly describe the short-time and the long-time behavior of the correlation function, the approximation given in (2.6) has to reproduce C_E $(t=0)$ and $\tilde{C}_E$ $(\omega=0)$, the latter being the lowest moment μ_{-1} of the low-frequency expansion (2.10), thereby leading to (2.8) for the relaxation times. Higher-order approximations could be obtained by assuming instead of (2.6) a series of exponentials which have to reproduce more moments of (2.10), and of the corresponding high-frequency expansion, see [2–7].

We note that the single-exponential form (2.6) of the approximate correlation function corresponds to an approximation of the spectral lineshape of the fluctuations by a single Lorentzian, i.e.

$$\mathrm{Re}\{\tilde{C}_j(i\omega)\} \approx \frac{C_j(0)}{\tau_j} \frac{1}{\omega^2+\tau_j^{-2}}, \quad j=I, E, \tag{2.12}$$

with the effective linewidth given by the inverse of the mean relaxation time τ_j. This effective linewidth is in a sense the *best* possible single-Lorentzian description of the exact spectral lineshape, since it represents an *interpolation* between the correct long-time (low-frequency) behavior and the correct short-time (high-frequency) behavior.

2.1. *Intensity fluctuations*

In order to consider the intensity fluctuations it is appropriate to transform the complex field E to its intensity I and phase φ through $E=\sqrt{I}\exp(i\varphi)$. This transformation leads to the following form for the adjoint Fokker-Planck operator

$$\mathbf{L}^+(I,\varphi)=\mathbf{L}_0^+(I)+\frac{1}{I}\frac{\partial^2}{\partial\varphi^2} \tag{2.13}$$

with the intensity part $\mathbf{L}_0^+(I)$ having the form

$$\mathbf{L}_0^+(I)=\left[\frac{\partial}{\partial I}+\frac{1}{2}(a-I)\,I\right]4I\frac{\partial}{\partial I}. \tag{2.14}$$

For the determination of the matrix element in (2.8a) the angular part in (2.13) is irrelevant. Therefore, the relaxation rate is determined solely by the one-dimensional adjoint Fokker-Planck operator (2.14). This operator is of Smoluchowski type [14] for which the adjoint has the general form

$$\mathbf{L}^+(I)=\left[\frac{\partial}{\partial I}-\left(\frac{d}{dI}\,U(I)\right)\right]D(I)\frac{\partial}{\partial I}. \tag{2.15}$$

Equation (2.15) describes a one-dimensional diffusion process in a potential $U(I)$ with a position dependent diffusion coefficient $D(I)$. In our case, the diffusion coefficient is given by $D(I)=4I$, and the potential $U(I)$ is given by (2.3), with $|E|^2$ replaced by I. In [18] we have shown that for an adjoint Fokker-Planck operator (2.15) the matrix element of the inverse operator, as it appears in (2.8a), can be determined by evaluating the integral

$$\langle \delta I[\mathbf{L}^+(I)]^{-1}\delta I\rangle = \int_0^\infty dI[D(I)\,p_0(I)]^{-1}\left|\int_0^I dI' p_0(I')\,\delta I'\right|^2. \tag{2.16}$$

By employing the representation

$$\int_0^I dI' p_0(I')\,\delta I' = \frac{a}{\mathrm{erf}\left(\frac{a}{2}\right)}\left[\mathrm{erf}\left(\frac{I-a}{2}\right)-1\right]-2p_0(I) \tag{2.17}$$

for the innermost integral and using numerical approximations [19] for the error function, (2.16) can be easily evaluated via numerical integration.

Figure 1 shows the results for the effective linewidth, i.e., the inverse of the mean relaxation time τ_I, obtained from a numerical evaluation of the integral (2.16). We note that our linewidth is identical to the effective linewidth considered by Risken and Vollmer [12, 14] given by

$$\frac{1}{\lambda}=\sum_{n=1}^{\infty}\frac{V_n}{\lambda_{0n}}, \tag{2.18}$$

where the λ_{0n} denote the (nonzero) eigenvalues of the intensity part (2.14) of the full Fokker-Planck opera-

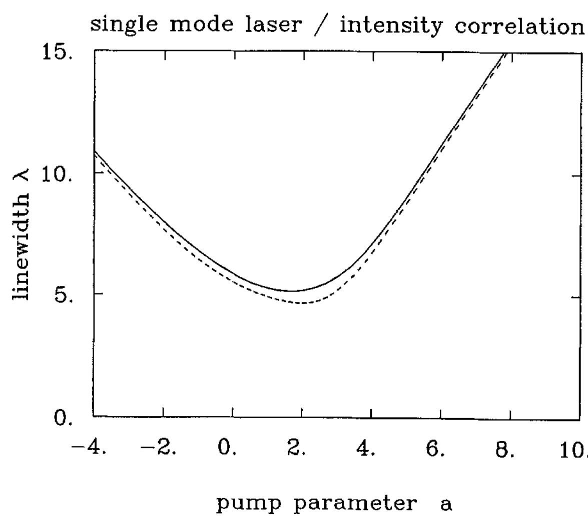

Fig. 1. Effective linewidth λ for the intensity fluctuations vs. pump-parameter a. (——) is our result from a numerical integration of (2.16); note that this is identical to Risken's result [12, 14]; (– – –) is the result of Grossmann's mode-coupling calculation [10]

tor (2.13), and the V_n denote the corresponding expansion coefficients of $C_I(t)$. λ has been determined by Risken and Vollmer [12, 14] by summing up according to (2.18) the lowest numerically determined eigenvalues and expansion coefficients. The MRTA gives a single analytical expression for this quantity, (2.16). For comparison we included in Fig. 1 the results obtained by Grossmann from a mode-coupling approximation [10]. This approach is seen to underestimate the effective linewidth.

We would like to note at this point that fluctuations of the pump parameter a, relevant in dye lasers [20–23], give rise to stochastic models featuring multiplicative noise. In the Appendix we show how the framework of the MRTA applies in such a situation.

2.2. Field amplitude fluctuations

In the following we write the complex field amplitude E as vector

$$\mathbf{E}=|E|\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}, \tag{2.19}$$

where the first component represents the real part, and the second component represents the imaginary part of E. In this representation the adjoint operator is

$$\mathbf{L}^+(\mathbf{E})=\mathbf{L}_0^+(|E|)+\frac{1}{|E|^2}\frac{\partial^2}{\partial\varphi^2}, \tag{2.20}$$

and the radial part $\mathbf{L}_0^+(|E|)$ can be written as

$$\mathbf{L}_0^+(|E|)=\frac{1}{|E|\,p_0(|E|)}\frac{\partial}{\partial|E|}|E|\,p_0(|E|)\frac{\partial}{\partial|E|}. \tag{2.21}$$

To determine the matrix element in (2.8b) we first define an auxiliary vector function $\boldsymbol{\mu}_{-1}(\mathbf{E})$, through

$$\boldsymbol{\mu}_{-1}(\mathbf{E})=-[\mathbf{L}^+(\mathbf{E})]^{-1}\delta\mathbf{E}. \tag{2.22}$$

$\boldsymbol{\mu}_{-1}(\mathbf{E})$ is, in effect, the right hand side function in the scalar product that contributes to the matrix element in (2.8b). This auxiliary function can be determined as the solution of the equation

$$\mathbf{L}^+(\mathbf{E})\,\boldsymbol{\mu}_{-1}(\mathbf{E})=-\delta\mathbf{E} \tag{2.23}$$

with reflective boundary conditions [7, 18]. We first make the ansatz

$$\boldsymbol{\mu}_{-1}(\mathbf{E})=\mu_{-1}(|E|)\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}. \tag{2.24}$$

This ansatz leads to the one-dimensional equation

$$\left[\mathbf{L}_0^+(|E|)-\frac{1}{|E|^2}\right]\mu_{-1}(|E|)=-|E| \tag{2.25}$$

for the radial $\mu_{-1}(|E|)$, where we have already taken into account that the average of E vanishes. Equation (2.25) has to be supplemented with reflective boundary conditions for $|E|=0$ and $|E|\to\infty$, see [7] and [18]. The operator on the left hand side of (2.25) corresponds to an adjoint one-dimensional Fokker-Planck operator with a *reactive term*. Such a term renders a general analytical solution of this equation impossible, and, due to its singular nature, methods for treating it in a perturbative way also fail. However, the differential Eq. (2.25) can be very easily solved numerically by discretizing the one-dimensional state space for $|E|$, employing the methods of [7]. In this discretization scheme the singular nature of the reaction term gives no problems since the singularity lies on the (lower) boundary of the state space. The resulting linear equation is of tridiagonal form and can, therefore, be directly solved using the Gaussian elimination scheme [19]. In an actual calculation one has to introduce an upper limit for the state space, and the independence of the numerical results from this upper limit, as well as from the actual value of the discretization length, have to be checked. The matrix element is finally determined from the auxiliary function $\mu_{-1}(|E|)$ through the integral

$$\begin{aligned}\mu_{-1}&=\langle\delta E^*[\mathbf{L}^+(E)]^{-1}\delta E\rangle\\ &=\int_0^{2\pi}\int_0^{\infty}|E|[\mathbf{E}\cdot\boldsymbol{\mu}_{-1}(\mathbf{E})]\,p_0(|E|)\,d|E|\,\mathrm{d}\varphi\\ &=2\pi\int_0^{\infty}|E|^2\mu^{-1}(|E|)\,p_0(|E|)\,\mathrm{d}|E|.\end{aligned} \tag{2.26}$$

In Fig. 2 our numerical results for the linewidth factor α are presented and comparison is made with

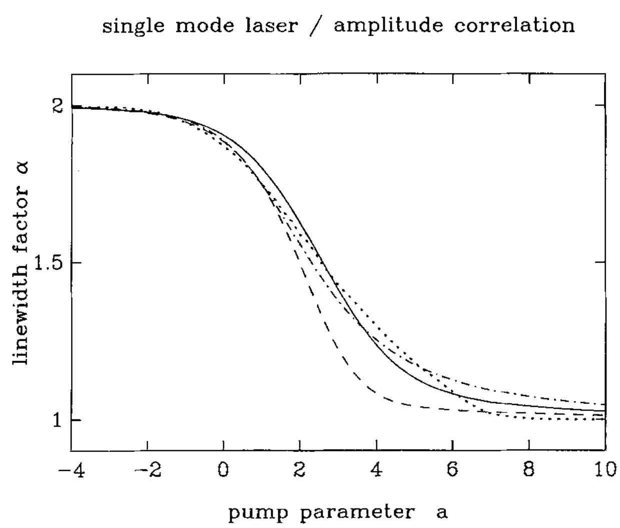

Fig. 2. Linewidth factor α for the field amplitude fluctuations vs. pump-parameter a. (——) is our result from a numerical solution of (2.25) and (2.26); (–––) is the result of Grossmann's mode coupling calculation [10]; (····) is the result of Risken's numerical approximation [11, 14]; (–·–) is the result of Ziegler and Horner's analytical approximation [24], (2.28)

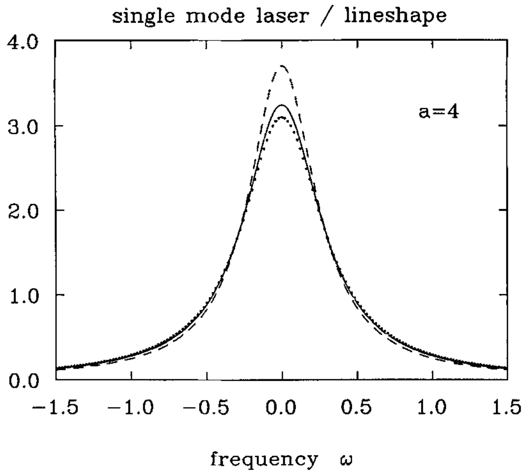

Fig. 3. Lineshape (2.12) of the field amplitude fluctuations resulting from the approximations in Fig. 2 for a pump-parameter $a=4$; the linetypes correspond to those in Fig. 2

the mode-coupling results by Grossmann [10], and with the results of the eigenvalue analysis by Risken [11, 14]. The linewidth factor is, in effect, the inverse of the mean relaxation time,

$$\alpha = \frac{\langle I \rangle}{\tau_E}. \tag{2.27}$$

The discrepancy between our results and the results of Risken stem from the fact that Risken considers only the lowest eigenvalue λ_{10} of the operator (2.20), since it gives the main contribution to the decay of $C_E(t)$. The mean relaxation time calculated by our method takes into account also the contributions from the other eigenvalues, as was already the case for the intensity fluctuations, see (2.18).

In order to evaluate how the discrepancy between those three different results is reflected in the corresponding lineshapes, (2.12), we represent them in Fig. 3 for a pump-parameter $a=4$. We choose this value of a since here the discrepancy between the three different results is relatively large. A comparison with a higher-order approximation based on the GME according to [2] and [7] shows that our MRTA description is indistinguishable from the exact lineshape. Risken's results are already very close to the actual lineshape, however our results represent an additional small improvement. It can be seen that Grossmann's mode-coupling results give an inaccurate description of the low-frequency part of the lineshape. This is no surprise since Grossmann's results are based on a high-frequency expansion, although refined by the mode-coupling approximation.

We note that Ziegler and Horner [24] have calculated selfconsistently a quantity that corresponds to our mean relaxation time, using a partial summation of a perturbation expansion (corresponding to a random phase approximation). Their analytical result for the linewidth factor

$$\alpha_{ZH} = 2 - \frac{1}{\left(1+\frac{2}{\langle I \rangle^2}\right)\left(1+\frac{3}{\langle I \rangle^2}\right)} \tag{2.28}$$

is compared with our numerical results also in Fig. 2. It can be seen that, although they differ from each other, Ziegler and Horner's analytical approximation and Risken's numerical approximation are about equally close to our numerical results.

3. Detuned single mode laser

If there is a mismatch between the frequency of the atomic transition in the laser medium and the frequency of the resonance cavity, the laser is detuned. This situation can be described by a modification of the Fokker-Planck operator (2.2) [13, 14, 25], its adjoint having the form

$$\mathbf{L}_d^+(E) = \left[\frac{\partial}{\partial E} + (1+i\delta)(a-|E|^2)\,E\right]\frac{\partial}{\partial E^*}. \tag{3.1}$$

δ is the detuning parameter which is proportional to the frequency difference between the atomic transition and the cavity frequency. For the stochastic system described by (3.1), the stationary distribution for the field amplitude E is still given by $p_0(E)$, (2.4), with the potential $U(E)$ given by (2.3). Therefore, the static properties of the detuned laser remain unchanged upon going from (2.2) to (3.1) [13, 14]. However, dynamic properties are changed, since the detuning induces an angular drift [26]. This angular drift can be seen by transforming again the complex field amplitude E to the vector $\mathbf{E}$ according to (2.19). In this representation the adjoint Fokker-Planck operator has the form

$$\mathbf{L}_d^+(\mathbf{E}) = \mathbf{L}_0^+(|E|) + \delta(a-|E|^2)\frac{\partial}{\partial\varphi} + \frac{1}{|E|^2}\frac{\partial^2}{\partial\varphi^2}, \tag{3.2}$$

where the part $\mathbf{L}_0^+(|E|)$ is given as before, (2.21). It turns out that the angular drift term in (3.2) does not affect the intensity correlation function [13, 14]. However, the behavior of the field amplitude correlation function, (2.5b), is changed, and one expects an oscillatory behavior in this correlation function, superimposed on the decay of the fluctuations.

Such an oscillatory component in the decay of correlation functions can only occur in stochastic systems with a state space of dimension $d>1$. Another simple example of a system that exhibits this behavior is an underdamped Brownian particle in a harmonic potential, as described by the Kramers equation for the position and velocity of the particle [14]. Clearly, the MRTA as defined in (2.6) cannot describe oscillatory behavior [27]. In the following we introduce an extension of the MRTA that accommodates the approximate description of this behavior of correlation functions in higher-dimensional stochastic systems.

3.1. Modified MRTA

The most simple approximation taking into account that oscillations are superimposed on a single-exponential decay of a correlation function is

$$C_E(t) \approx C_E(0)\exp(-t/\tau_E)\cos(\Omega_E t). \tag{3.3}$$

In the spirit of the MRTA, the approximation (3.3) should describe correctly the long-time, i.e., low-frequency behavior of the exact correlation function. Therefore, we require (3.3) to reproduce the first two moments μ_{-1} and μ_{-2} of the low-frequency expansion (2.10). Taking into account the representation of these moments as matrix elements of the inverse adjoint Fokker-Planck operator, (2.11), this leads to the following equations for the parameters τ_E and Ω_E of approximation (3.3):

$$\begin{aligned}\tau_E &= (1+\varepsilon)\,\mu_{-1}/C_E(0)\\ &= -(1+\varepsilon)\langle \delta E^*[\mathbf{L}^+(E)]^{-1}\rangle/C_E(0)\end{aligned} \tag{3.4a}$$

and

$$\Omega_E = \frac{\sqrt{\varepsilon}}{\tau_E}, \tag{3.4b}$$

with

$$\varepsilon = 1 - C_E(0)\frac{\mu_{-2}}{\mu_{-1}^2} = 1 - C_E(0)\frac{\langle \delta E^*[\mathbf{L}^+(E)]^{-2}\delta E\rangle}{\langle \delta E^*[\mathbf{L}^+(E)]^{-1}\delta E\rangle^2}. \tag{3.4c}$$

We note that for $\varepsilon \to 0$ the mean relaxation time τ_E assumes the old form, (2.8b). In the same limit Ω_E vanishes. The parameter ε is, therefore, a measure for the deviation of the behavior of the exact correlation function from a single-exponential decay. In the modified MRTA it is assumed that this deviation arises solely from a superimposed oscillation. We will see that for the field amplitude correlations of the detuned single mode laser this assumption is justified.

A negative value of ε would give rise to an imaginary frequency Ω_E, resulting in a purely relaxational behavior of the approximate correlation function, (3.3). Such a result would indicate that the dominant behavior of the correlation function is not oscillatory. In fact, by a spectral expansion of the correlation function, cf. part I, one can prove that ε is negative if the correlation function has no oscillatory component. Only if a pronounced oscillatory component is present in the spectral expansion can the value of ε become positive.

A higher-order extension of this modified MRTA would be the use of a series of exponentials with complex relaxation times, the parameters determined according to the GME principles [2, 7], i.e., reproducing a certain number of generalized moments. Such an extension may also be employed to check the quality of the modified MRTA.

3.2. Field amplitude fluctuations

We have seen in the previous section that, in addition to the low-frequency moment μ_{-1}, the second moment μ_{-2} is necessary to describe the behavior of the field amplitude fluctuations of the detuned single mode laser with the modified MRTA. In order to calculate both matrix elements, we introduce, similarly to Sect. 2.2, the auxiliary vector functions

$$\boldsymbol{\mu}_{-n}(\mathbf{E}) = (-1)^n[\mathbf{L}_d^+(\mathbf{E})]^{-n}\mathbf{E} \quad \text{for } n=1,2. \tag{3.5}$$

They are given as solutions of the equations

$$\mathbf{L}_d^+(\mathbf{E})\,\boldsymbol{\mu}_{-n}(\mathbf{E}) = -\boldsymbol{\mu}_{-(n-1)}(\mathbf{E}) \quad \text{for } n=1,2, \tag{3.6}$$

with $\boldsymbol{\mu}_0(\mathbf{E}) = \mathbf{E}$ as right hand side function of the first equation, and reflective boundary conditions [7, 18]. With the ansatz

$$\boldsymbol{\mu}_{-n}(\mathbf{E}) = \begin{pmatrix} \mu_{-n}^{(11)}(|E|) & \mu_{-n}^{(12)}(|E|) \\ \mu_{-n}^{(21)}(|E|) & \mu_{-n}^{(22)}(|E|) \end{pmatrix}\begin{pmatrix}\cos\varphi \\ \sin\varphi\end{pmatrix} \quad \text{for } n=1,2, \tag{3.7}$$

we finally arrive at the set of equations

$$\begin{pmatrix} \mathbf{L}_0^+(|E|) - \frac{1}{|E|^2} & (a-|E|^2)\delta \\ -(\alpha-|E|^2)\delta & \mathbf{L}_0^+(|E|) - \frac{1}{|E|^2} \end{pmatrix}\begin{pmatrix}\mu_{-1}^{(11)}(|E|) \\ \mu_{-1}^{(12)}(|E|)\end{pmatrix} = -\begin{pmatrix}|E| \\ 0\end{pmatrix} \tag{3.8a}$$

and

$$\begin{pmatrix} \mathbf{L}_0^+(|E|) - \frac{1}{|E|^2} & (a-|E|^2)\delta \\ -(a-|E|^2)\delta & \mathbf{L}_0^+(|E|) - \frac{1}{|E|^2} \end{pmatrix}\begin{pmatrix}\mu_{-2}^{(11)}(|E|) \\ \mu_{-2}^{(12)}(|E|)\end{pmatrix} = -\begin{pmatrix}\mu_{-1}^{(11)}(|E|) \\ \mu_{-1}^{(12)}(|E|)\end{pmatrix}. \tag{3.8b}$$

Again, reflective boundary conditions holds for the scalar functions $\mu_{-n}^{(ij)}(|E|)$ for $|E|=0$ and $|E|\to\infty$. Since the same set of equations as in (3.8) holds for the functions $\mu_{-n}^{(22)}(|E|)$ and $\mu_{-n}^{(21)}(|E|)$, they are equal to $\mu_{-n}^{(11)}(|E|)$ and $\mu_{-n}^{(12)}(|E|)$, respectively. Therefore, only the above equations have to be considered.

Equations (3.8) bear strong similarities to (2.25) and can again be solved numerically by discretization of the operator $\left[\mathbf{L}_0^+(|E|)-\frac{1}{|E|^2}\right]$ according to [7]. In contrast to the case in Sect. 2.2, here the discretization does not lead to a simple tri-diagonal form of the resulting linear equations, due to the off-diagonal elements being $\propto\delta$. However, the resulting matrix still has a simple band structure. The linear equations resulting from (3.8) can, therefore, be solved either be rearrangement into a band-diagonal structure and employing numerical routines for such a situation [19], or by employing *sparse matrix techniques* (see [7] and references cited therein) directly. We chose the latter way in our calculations.

The matrix elements are determined, finally, from the auxiliary functions $\mu_{-n}^{(ij)}(|E|)$ through integration

$$\begin{aligned}\mu_{-n}&=\langle\delta E^*[\mathbf{L}_d^+(E)]^{-n}\delta E\rangle\\&=\int_0^{2\pi}\int_0^{\infty}|E|[\mathbf{E}\cdot\boldsymbol{\mu}_{-n}(\mathbf{E})]\,p_0(|E|)\,\mathrm{d}|E|\,\mathrm{d}\varphi\\&=2\pi\int_0^{\infty}|E|^2\mu_{-n}^{(11)}(|E|)\,p_0(|E|)\,\mathrm{d}|E|\qquad\text{for } n=1,2.\end{aligned}\tag{3.9}$$

Figure 4a shows our numerical results for the linewidth factor α, (2.27), and Fig. 4b shows the results for the frequency shift Ω_E, both for a value of the detuning parameter δ of unity. For sake of comparison with the case of no detuning, our results from Sect. 2.2 are included in Fig. 4a ($\delta=0$). These results are compared to the results by Seybold and Risken [13, 14] for the real and imaginary part of the lowest nonzero eigenvalue of $\mathbf{L}_d(\mathbf{E})$, respectively. The deviations are in general quite small, thereby justifying the assertion in [13] that eigenvalues other than the lowest one give only minor contributions to the effective behavior of the field amplitude fluctuations.

Again, we have checked the corresponding lineshapes. Figure 5 shows a comparison of the lineshapes for a pump-parameter $a=4$, and a detuning parameter $\delta=1$. As was the case in Sect. 2, a higher-order approximation shows that our MRTA description of the lineshape is already indistinguishable from

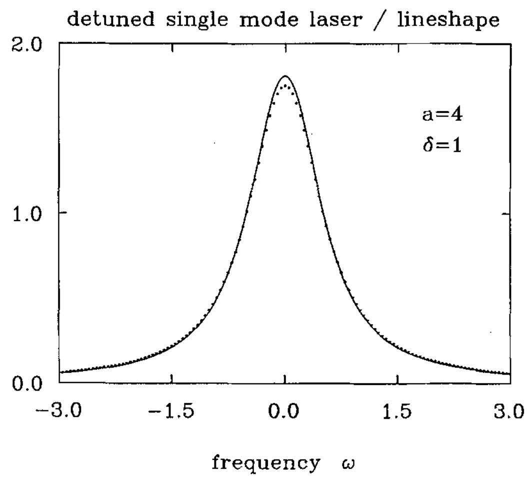

Fig. 5. Lineshape (2.12) of the field amplitude fluctuations resulting from the approximations in Fig. 4 for a pump-parameter $a=4$ and a detuning-parameter $\delta=1$; the linetypes correspond to those in Fig. 4

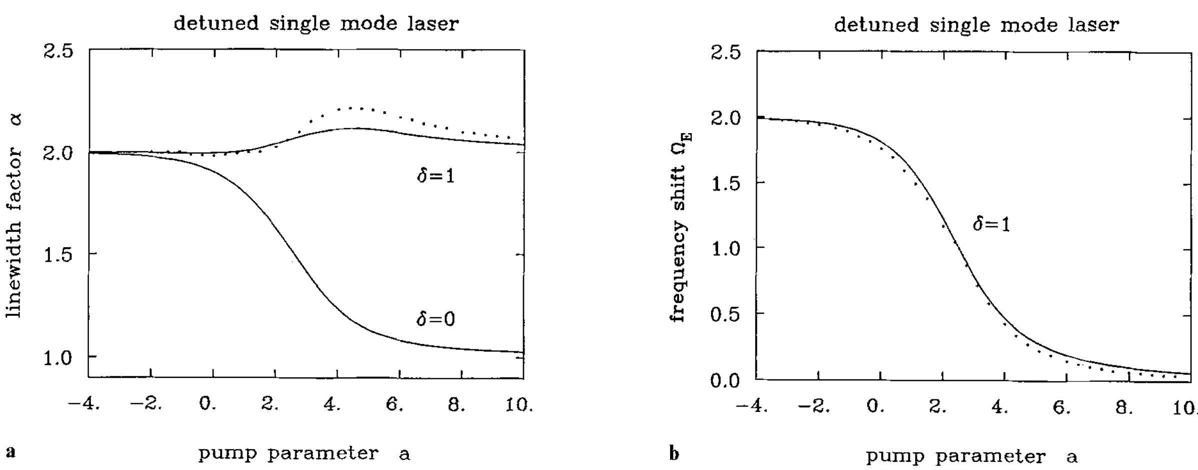

Fig. 4a and b. Linewidth factor α (**a**) and frequency shift Ω_E (**b**) for the field amplitude fluctuations vs. pump-parameter a in the detuned laser for values of the detuning parameter $\delta=0$, 1. (——) is our result from a numerical solution of (3.8) and (3.9) and an extended MRTA according to Sect. 3.1; (····) is the result of Seybold and Risken [13, 14] the exact lineshape. Seybold and Risken's results are very close to the actual lineshape, however our results represent again an additional small improvement.

4. Discussion

We have demonstrated in this contribution the application of the MRTA to correlation functions in a generic two-dimensional stochastic system, the stochastic description of the single mode laser. It was shown that already the MRTA-description of the spectral lineshape of the fluctuations is indistinguishable from the exact lineshape. The quality of the effective linewidths determined from this approach is, therefore, slightly better than the one derived from an eigenvalue analysis by taking the lowest eigenvalue. In our approach, the effective linewidth can be calculated either using analytical expressions in the form of quadratures, (2.16), or by solving a single system of linear equations (two coupled equations in the case of the detuned laser). We, therefore, consider this method to be somewhat simpler numerically and more effective than approaches based on an analysis of the lowest eigenvalues.

An extension of our approach to systems with still more complicated higher-dimensional state space is readily possible [28]. Models for multi-mode lasers [29–33], and models that describe effects of additive and multiplicative colored noise in lasers [34–38] are particularly interesting stochastic systems in this context. The theoretical analysis of slow fluctuations in such systems has up to now concentrated on the determination of first passage times for the switching between different stable states, or on the determination of the lowest eigenvalue. With the MRTA and its higher-order extension, the GME, a more refined analysis, in particular taking into account the specific properties of the different correlation functions of interest, would be possible. In addition, using the MRTA for correlation functions in bistable colored noise systems would give a simpler approach to the problem of switching times in those systems, since it removes mathematical problems associated with the boundary conditions for first passage time problems [37].

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Appendix: MRTA for 1 d stochastic models with multiplicative white noise

The influence of fluctuations of the pump-parameter on the laser process, e.g. relevant in dye lasers, can be described by stochastic models featuring *multiplicative noise* [20–23]. The starting point of such a description is a (formal) nonlinear Langevin equation, e.g. for the intensity I, of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) = h(I) + g(I)\,\xi(t). \tag{A1}$$

$h(I)$ and $g(I)$ are functions depending on the model employed, and $\xi(t)$ is a Gaussian noise with a white spectrum. In the Stratonovitch interpretation [14] (A 1) is equivalent to a Fokker-Planck equation with a Fokker Planck operator

$$\mathbf{L}(I) = -\frac{\partial}{\partial I}[h(I) + g(I)\,g'(I)] + \frac{\partial^2}{\partial I^2} g^2(I). \tag{A2}$$

This operator can be easily brought into Smoluchowski form,

$$\mathbf{L}(I) = \frac{\partial}{\partial I} D(I)\left[\frac{\partial}{\partial I} + \left(\frac{\mathrm{d}}{\mathrm{d}I} U(I)\right)\right], \tag{A3}$$

describing a diffusion process in the potential $U(I)$ with the position-dependent diffusion coefficient $D(I)$. In our case these quantities are

$$D(I) = g^2(I), \tag{A4}$$

$$U(I) = \ln g(I) - \int^{I} \mathrm{d}I'\, h(I')/g^2(I'). \tag{A5}$$

The stationary distribution following from operator (A 2) is the Boltzmann distribution $p_0(I) \propto \exp[-U(I)]$.

Employing the form (A 3) for the Fokker-Planck operator, correlation functions in such systems can now be analyzed along the same lines as we have done in the main part of the present paper. In particular, for the autocorrelation function of the intensity fluctuations,

$$C_I(t) = \langle \delta I(0)\,\delta I(t)\rangle \approx C_I(0)\exp(-t/\tau_I), \tag{A6}$$

the mean relaxation time τ_E is again given by an integral expression, see Sect. 2.1,

$$\begin{aligned}\tau_I &= -\langle \delta I[\mathbf{L}^+(I)]^{-1}\delta I\rangle / C_I(0)\\ &= \int_0^\infty \mathrm{d}I[D(I)\,p_0(I)]^{-1}\left|\int_0^I \mathrm{d}I'\,p_0(I)\right|^2 \Big/ C_I(0).\end{aligned} \tag{A7}$$

In addition, the low-frequency moments necessary for higher-order approximations can be readily determined by applying the integral expressions we derived in [18].

Mean relaxation times for correlation functions in two stochastic models with multiplicative white noise, the Verhulst model and the Stratonovitch model, have recently been determined by Jung and Risken [22], employing the integral expression in (A7) which they derived independently. Their numerical analysis indicates that in those models an MRTA is sufficient for the description of the correlation functions only for weak noise. For larger values of the noise-parameter the correlation functions exhibit a more complicated behavior, which can probably be described by generalizing the MRTA (A6) to a higher-order approximation, consisting of two or three exponentials.

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